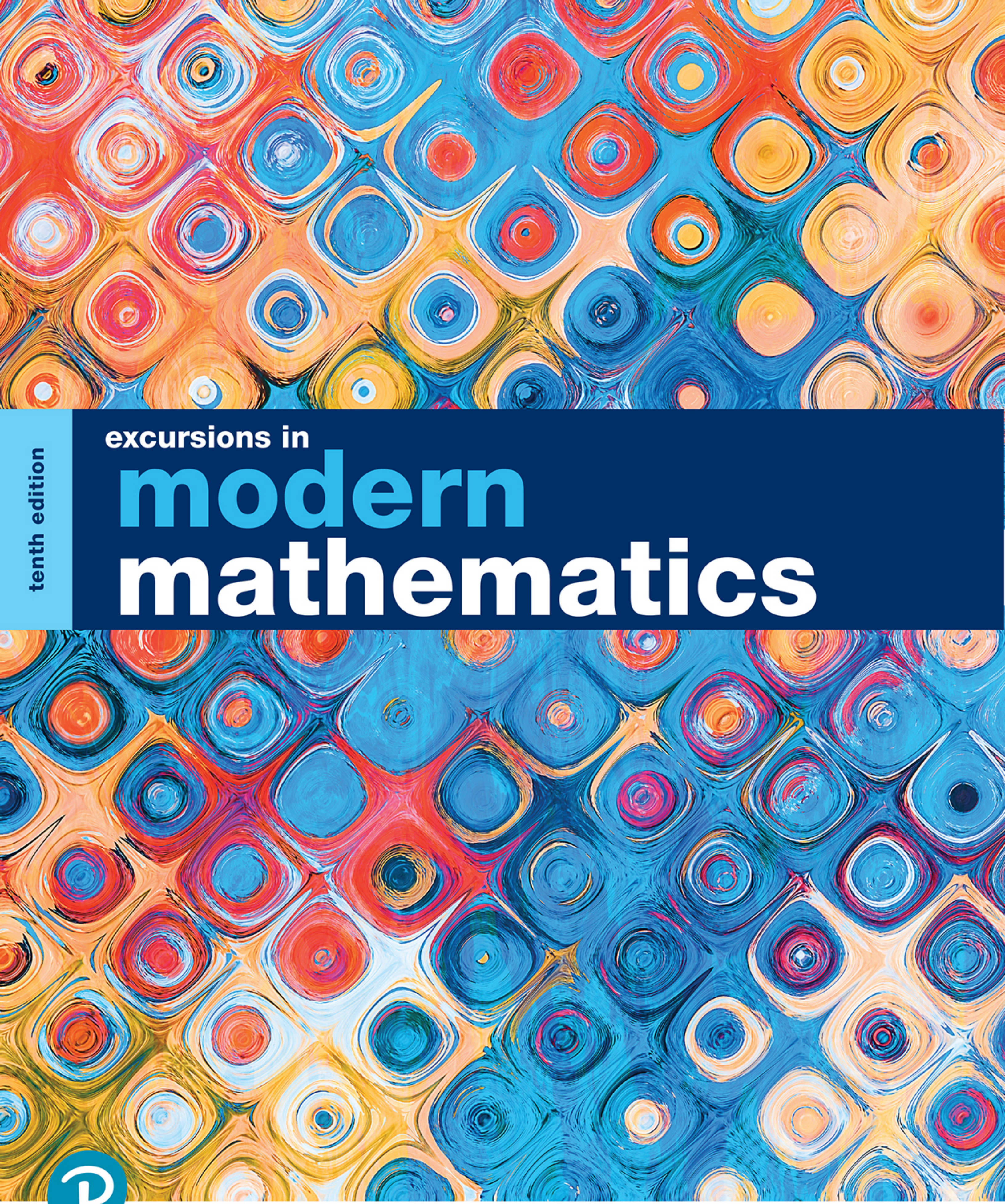

excursions in

# modern mathematics

tenth edition

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# Excursions in Modern Mathematics

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TENTH EDITION

# Excursions in Modern Mathematics

Peter Tannenbaum
California State University—Fresno

Pearson

Content Development: *Robert Carroll*
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*To the members of the board of Last Tango*

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# Contents

# PART 4 Shape and Form

# PART 5 Statistics

# Preface

> "To most outsiders, modern mathematics is unknown territory. Its borders are protected by dense thickets of technical terms; its landscapes are a mass of indecipherable equations and incomprehensible concepts. Few realize that the world of modern mathematics is rich with vivid images and provocative ideas."
>
> – *Ivars Peterson,*
> The Mathematical Tourist

This text started many years ago as a set of lecture notes for a new, experimental "math appreciation" course (these types of courses are described, sometimes a bit derisively, as "math for poets"). Over time, the lecture notes grew into a text and the "poets" turned out to be social scientists, political scientists, economists, psychologists, environmentalists, and many other "ists." Over time, and with the input of many users, the contents have been expanded and improved, but the underlying philosophy of the text has remained the same since those handwritten lecture notes were handed out to my first group of students.

***Excursions in Modern Mathematics*** is a travelogue into that vast and alien frontier that many people perceive mathematics to be. My goal is to show the open-minded reader that mathematics is a lively, interesting, useful, and surprisingly rich human activity.

The "excursions" in *Excursions* represent a collection of topics chosen to meet the following simple criteria.

- **Applicability.** There is no need to worry here about that great existential question of college mathematics: What is this stuff good for? The connection between the mathematics presented in these excursions and down-to-earth, concrete real-life problems is transparent and immediate.
- **Accessibility.** As a general rule, the excursions in this text do not presume a background beyond standard high school mathematics—by and large, intermediate algebra and a little Euclidean geometry are appropriate and sufficient prerequisites. (In the few instances in which more advanced concepts are unavoidable, an effort has been made to provide enough background to make the material self-contained.) A word of caution—this does not mean that the excursions in this book are easy! In mathematics, as in many other walks of life, simple and basic are not synonymous with easy and superficial.
- **Modernity.** Unlike much of traditional mathematics, which is often hundreds of years old, most of the mathematics in this text has been discovered within the last 100 years, and in some cases, within the last couple of decades. Modern mathematical discoveries do not have to be the exclusive province of professional mathematicians.
- **Aesthetics.** The notion that there is such a thing as beauty in mathematics is surprising to most casual observers. There is an important aesthetic component in mathematics, and just as in art and music (which mathematics very much resembles), it often surfaces in the simplest ideas. A fundamental objective of this text is to develop an appreciation of the aesthetic elements of mathematics.

## Outline of Contents

The excursions are organized into five independent parts, each touching on a different area where mathematics and the real-world interface.

**PART 1 Social Choice.** This part deals with mathematical applications to politics, social science, and government. How are *elections* decided? (Chapter 1); How can the power of individuals, groups, or voting blocs be measured? (Chapter 2); How can assets commonly owned be *divided* in a *fair* and equitable manner? (Chapter 3); How are seats *apportioned* in a legislative body? (Chapter 4).

**PART 2 Management Science.** This part deals with questions of efficiency—how to manage some valuable resource (time, money, energy) so that utility is maximized. How do we sweep over a network with the least amount of backtracking? (Chapter 5); How do we find the shortest or least expensive route that *visits* a specified set of locations? (Chapter 6); How do we create efficient networks that *connect* people or things? (Chapter 7); How do we schedule a project so that it is completed as early as possible? (Chapter 8).

**PART 3 Growth.** In this part, we discuss, in very broad terms, the mathematics of growth and decay, profit and loss. In Chapter 9, we cover mathematical models of *population growth*, mostly biological and human populations but also populations of inanimate "things" such as garbage and pollution. Since money plays such an important role in our lives, it deserves a chapter of its own. In Chapter 10, we discuss a few of the key concepts of *financial mathematics*: interest, investments, retirement savings, and consumer debt.

**PART 4 Shape and Form.** In this part, we cover a few connections between mathematics and the shape and form of objects—natural or human-made. What is *symmetry*? What *types* of symmetries exist in nature and art? (Chapter 11); What kind of geometry lies hidden behind the *kinkiness* of the many irregular shapes we find in nature? (Chapter 12); What is the connection between the *Fibonacci numbers* and the *golden ratio* (two abstract mathematical constructs) and the *spiral* forms that we regularly find in nature? (Chapter 13).

**PART 5 Statistics.** In one way or another, statistics affects all our lives. Government policy, insurance rates, our health, our diet, and our political lives are all governed by statistical information. This part deals with how the statistical information that affects our lives is collected, organized, and interpreted. What are the purposes and strategies of *data collection*? (Chapter 14); How is data *organized, presented,* and *summarized*? (Chapter 15); How do we use mathematics to measure *uncertainty* and *risk*? (Chapter 16); How do we use mathematics to model, analyze, and make predictions about *real-life, bell-shaped* data sets? (Chapter 17).

## Exercise Sets

An important goal for this book is that it be flexible enough to appeal to a wide range of readers in a variety of settings. The exercise sets at the end of each chapter have been designed to convey the depth of the subject matter and are organized by level of difficulty:

- **Walking.** These exercises are meant to test a basic understanding of the main concepts, and they are intended to be within the capabilities of students at all levels.
- **Jogging.** These are exercises that can no longer be considered as routine–either because they use basic concepts at a higher level of complexity or they require slightly higher-order critical thinking skills, or both.
- **Running.** This is an umbrella category for problems that range from slightly unusual or slightly above average in difficulty to problems that can be a real challenge to even the most talented of students.
- **Applet Bytes.** Some chapters include at the end of the exercise set a set of *Applet Byte* exercises. These are exercises that involve the use of one of the applets that are available to accompany this text. The applets are available to all students, either through the MyLab course or by following the link *bit.ly/2NcwKFn*.

# New to This Edition

- New and updated examples from pop culture, sports, politics, and science keep the discussion current and relevant for today's students. Examples include discussion of the COVID-19 pandemic, the role of vaccines, and election polling.
- New and updated exercises have been informed by MyLab Math data analytics.
- Many MyLab exercises have been redesigned to more closely match the text's pedagogy. In many cases, this includes updating the learning aids, such as "Help Me Solve This" and "View an Example," to provide students a consistent experience between the text and the MyLab materials.
- New and updated videos have been added to the MyLab course to pair with the examples in the text, featuring expanded example video coverage new to this edition. When needed, videos in the MyLab have been updated to match any updates made to examples in the text.
- New StatCrunch data sets have been added, which allow users to see the full data behind some examples and exercises used in the book. These data sets are identified in the text with a StatCrunch icon **StatCrunch**. These data sets can be used and manipulated by students to better understand the relevant concepts and ideas and answer questions about them.
- **Integrated Review** content and assessments are now available in the MyLab course. Integrated Review assessments, provided for each chapter, allow the user to diagnose gaps in prerequisite skills that would impede progress on course-level objectives. Users can then use *personalized* homework assignments to address any gaps in skills identified. With personalized assignments, each user works on only those skills that they have not mastered.
- **Personal Inventory Assessments**, located in the Skills for Success module in the MyLab, are a collection of exercises designed to promote self-reflection and engagement in students. These 33 assessments include topics such as a Stress Management Assessment, Diagnosing Poor Performance and Enhancing Motivation, and Time Management Assessment.
- A list of the MyLab resources available for each section can now be found at the end of each chapter in the Annotated Instructor's Edition.

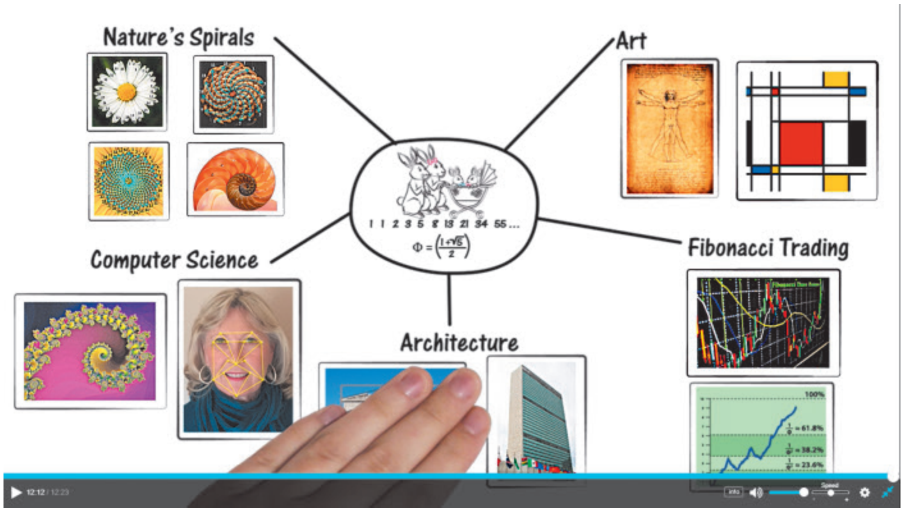
# MyLab® Math

MyLab Math is an online course delivery and course management platform that is integrated with this text. The MyLab resources can be used either to complement the text or for a stand-alone course, and include the following:

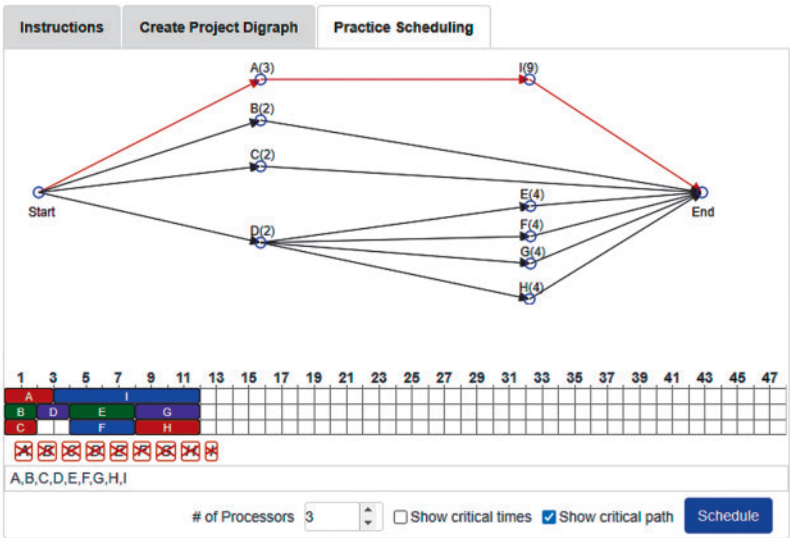
## Student Resources

- **eText**—Available in two formats: one that matches the textbook page-for-page, and another that is "reflowable" for use on tablets and smartphones. The latter eText is also fully accessible using screen-readers.
- **UPDATED! Exercises with Immediate Feedback**—The exercises in MyLab Math reflect the approach and learning style of this text and regenerate algorithmically to give students unlimited opportunity for practice and mastery. Most exercises include learning aids, such as guided solutions and sample problems, and they offer helpful feedback when students enter incorrect answers. The exercises are parallel to the exercises in the text, cover all levels of difficulty and include some Applet-based exercises.

- **UPDATED! Example Videos**—Example videos cover many of the examples in the text to demonstrate the concepts through the voice of an instructor. All videos have closed captioning available.
- **Animated Whiteboard Concept Videos**—These videos use narration and animated drawing to bring concepts to life in an engaging manner making the concepts easier to comprehend. Videos cover topics such as Fair Division, Eulerizing Graphs, Self-Similarity, The Golden Ratio, and Normal Curves. These videos are also linked in the eText.




- **Personalized Homework**—With Personalized Homework activated for an assignment, students taking the quiz or test receive a subsequent homework assignment that is personalized based on their performance. This way, students can focus on just the topics they have not yet mastered.
- **Applets**—These applets found in the Video & Resource Library and Learning Tools help students explore concepts more deeply, encouraging them to visualize and interact with concepts such as apportionment, methods, Hamilton paths and circuits, priority list scheduling, and geometric fractals. Applets are also linked in the eText.




- **Student's Solutions Manual** provides detailed worked out solutions to odd-numbered walking and jogging exercises. Instructors can choose to make this available to their class in the MyLab.

- **Projects & Papers**—The Projects & Papers included in earlier editions of the text are included as a MyLab Math resource for use as discussion material or project ideas.
- **Profiles**—The biographical profiles included in earlier editions of the text are also included as a MyLab Math for use as discussion material or project ideas.
- **NEW! StatCrunch** data sets have been added, which allow users to see the full data behind some examples and exercises used in the book.
- **Mindset videos** and assignable, open-ended exercises foster a growth mindset in students. This material encourages students to maintain a positive attitude about learning, value their own ability to grow, and view mistakes as learning opportunities–so often a hurdle for math students. These videos are one of many **Study Skills and Career-Readiness Resources** that address the nonmath-related issues that can affect student success.

## Instructor Resources

- **NEW! Integrated Review** in MyLab Math provides embedded and personalized review of prerequisite topics within relevant chapters. Integrated Review assignments, noted below, are premade and can be edited and assigned by instructors.
  - Students begin relevant chapters with a premade, assignable Skills Check to check each student's understanding of prerequisite skills needed to be successful in that chapter.
  - For any gaps in skill that are identified, a personalized review homework is populated. Students practice on the topics they need to focus on—no more, no less.
  - A suite of resources is available to help students understand the objectives they missed on the Skills Check quiz, including worksheets and videos to help remediate.
- Integrated Review in the MyLab is ideal for corequisite courses, where students are enrolled in a college-level course while receiving just-in-time remediation. But it can also be used simply to get underprepared students up to speed on prerequisite skills in order to be more successful in the main course content.
- **TestGen®** enables instructors to build, edit, print, and administer tests using a computerized bank of questions developed to cover all the objectives of the text.
- **Learning Catalytics**—Integrated into MyLab Math, Learning Catalytics (LC) uses students' mobile devices for an engagement, assessment, and classroom intelligence system that gives instructors real-time feedback on student learning. **LC** annotations in the Annotated Instructor's Edition provide a corresponding tag to search for when a LC question is relevant to the topic at hand. For more information on how to use these tags, go to bit.ly/3m0FYEB.
- **Instructor's Testing Manual** includes two alternative multiple-choice tests per chapter.
- **Instructor's Solutions Manual** contains detailed, worked out solutions to all exercises in the text.
- **Image Resources Library** contains all art from the text for instructors to use in their own presentations and handouts.
- **PowerPoint®** editable slides present key concepts and definitions from the text. You can add art from the Image Resource Library or slides that you develop on your own.
- **NEW! Early Alerts**—Now included with Performance Analytics, Early Alerts use predictive analytics to identify struggling students—even if their assignment scores are not a cause for concern. In both Performance Analytics and Early Alerts, instructors can e-mail students individually or by group to provide feedback.

**Available in print for instructors:**

- **REVISED! Annotated Instructor's Edition** (ISBN: 978-0-13-696895-5) provides annotations for instructors, including suggestions about where media resources like Applets and Animated Whiteboard Videos apply, as well as Learning Catalytics questions, discussion ideas, and teaching tips.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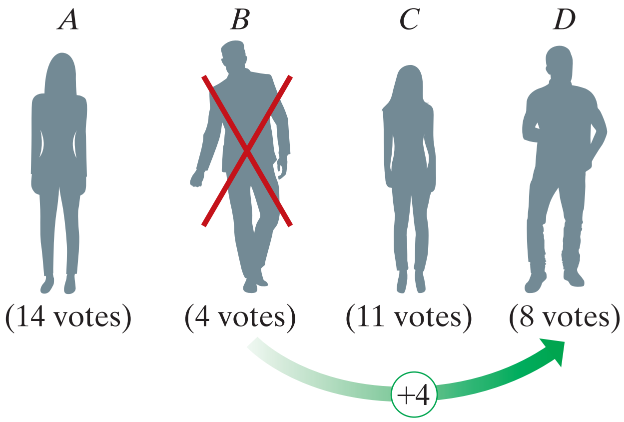
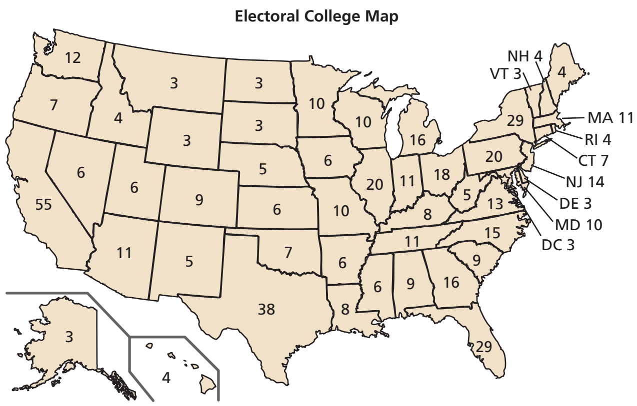
PART 1

# Social Choice

## 1 The Mathematics of Elections




## 2 The Mathematics of Power

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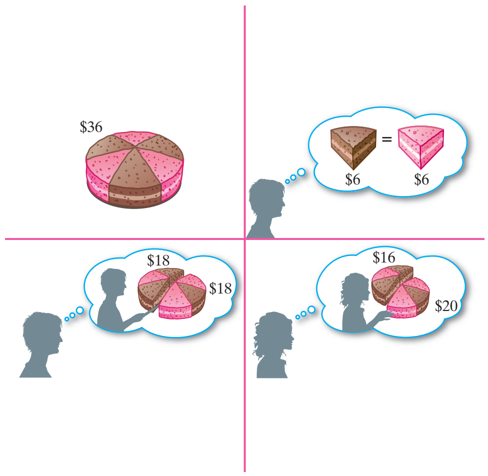
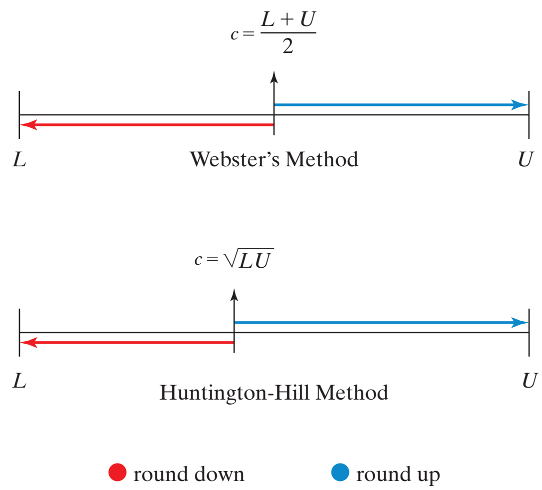



## 4 The Mathematics of Apportionment



1


△ 2019 *American Idol* top ten finalists (see Examples 1.5 and 1.16 for more).

# The Mathematics of Elections

## The Paradoxes of Democracy

Whether we like it or not, we are all affected by the outcomes of elections. Our president, senators, governors, and mayors make decisions that impact our lives in significant ways, and they all get to be in that position because an election made it possible. But elections touch our lives not just in politics. The Academy Awards, Heisman trophies, NCAA football rankings, *American Idol*—they are all decided by some sort of election. Even something as simple as deciding where to go for dinner might require a little family election.

We have elections because we don't all think alike. Since we cannot all have things our way, we vote. But *voting* is only the first half of the story, the one we are most familiar with. As playwright Tom Stoppard suggests, it's the second half of the story—the *counting*—that is at the heart of the democratic process. How do we sift through the many choices of individual voters to find the collective choice of the group? More important, how well does the process work? Is the process always fair? Answering these questions and explaining a few of the many intricacies and subtleties of *voting theory* are the purpose of this chapter.

But wait just a second! Voting theory? Why do we need a fancy theory to figure out how to count the votes? It all sounds pretty simple: We have an election; we count the ballots. Based on that count, we decide the outcome of the election in a consistent and fair manner. Surely, there must be a reasonable way to accomplish this. Surprisingly, there isn't!

In the late 1940s the American economist Kenneth Arrow discovered a remarkable fact: For elections involving three or more candidates, there is no consistently fair democratic method for choosing a winner. In fact, Arrow demonstrated that *a method for determining election results that is always fair is a mathematical impossibility*. This fact, the most famous in voting theory, is known as *Arrow's Impossibility Theorem*.

> "It's not the voting that's democracy; it's the counting."
>
> – *Tom Stoppard*

This chapter is organized as follows. We will start with a general discussion of *elections* and *ballots* in Section 1.1. This discussion provides the backdrop for the remaining sections, which are the heart of the chapter. In Sections 1.2 through 1.5 we will explore four of the most commonly used *voting methods*—how they work and how they are used in real-life applications. In Section 1.6 we will introduce some basic principles of fairness for voting methods and apply these *fairness criteria* to the voting methods discussed in Sections 1.2 through 1.5. The section concludes with a discussion of the meaning and significance of Arrow's Impossibility Theorem.

## 1.1 The Basic Elements of an Election

Big or small, important or trivial, *all* elections share a common set of elements.

- **The candidates.** The purpose of an election is to choose from a set of *candidates* or *alternatives* (at least two—otherwise it is not a real election). Typically, the word *candidate* is used for people and the word *alternative* is used for other things (movies, football teams, pizza toppings, etc.), but it is acceptable to use the two terms interchangeably. In the case of a generic choice (when we don't know if we are referring to a person or a thing), we will use the term *candidate*. While in theory there is no upper limit on the number of candidates, for most elections (in particular the ones we will discuss in this chapter) the number of candidates is small.
- **The voters.** These are the people who get a say in the outcome of the election. In most democratic elections the presumption is that all voters have an equal say, and we will assume this to be the case in this chapter. (This is not always true, as we will see in great detail in Chapter 2.) The number of voters in an election can range from very small (as few as 3 or 4) to very large (hundreds of millions). In this section we will see examples of both.
- **The ballots.** A ballot is the device by means of which a voter gets to express his or her opinion of the candidates. The most common type is a paper ballot, but a

voice vote, a text message, or an online vote can also serve as a "ballot" (see Example 1.5 *American Idol*). There are many different forms of ballots that can be used in an election, and Fig. 1-1 shows a few common examples. The simplest form is the **single-choice ballot**, shown in Fig. 1-1(a). Here very little is being asked of the voter ("pick the candidate you like best, and keep the rest of your opinions to yourself!"). At the other end of the spectrum is the **preference ballot**, where the voter is asked to rank *all* the candidates in order of preference. Figure 1-1(b) shows a typical preference ballot in an election with five candidates. In this ballot, the voter has entered the candidates' names in order of preference. An alternative version of the same preference ballot is shown in Fig. 1-1(c). Here the names of the candidates are already printed on the ballot and the voter simply has to mark first, second, third, etc. In elections where there are a large number of candidates, a **truncated preference ballot** is often used. In a truncated preference ballot the voter is asked to rank some, but not all, of the candidates. Figure 1-1(d) shows a truncated preference ballot for an election with dozens of candidates.

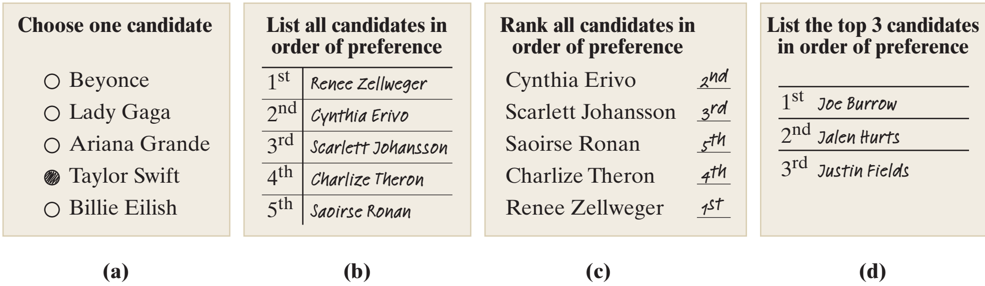



**Figure 1-1** (a) Single-choice ballot, (b) preference ballot, (c) a different version of the same preference ballot, and (d) truncated preference ballot.

- **The outcome.** The purpose of an election is to use the information provided by the ballots to produce some type of outcome. But what types of outcomes are possible? The most common is **winner-only**. As the name indicates, in a winner-only election all we want is to find a winner. We don't distinguish among the nonwinners. There are, however, situations where we want a broader outcome than just a winner—say we want to determine a first-place, second-place, and third-place candidate from a set of many candidates (but we don't care about fourth place, fifth place, etc.). We call this type of outcome a **partial ranking**. Finally, there are some situations where we want to rank *all* the candidates in order: first, second, third, . . . , last. We call this type of outcome a **full ranking**, or just a **ranking** for short.
- **The voting method.** The final piece of the puzzle is the method that we use to tabulate the ballots and produce the outcome. This is the most interesting (and complicated) part of the story, but we will not dwell on the topic here, as we will discuss voting methods throughout the rest of the chapter.

It is now time to illustrate and clarify the above concepts with some examples. We start with a simple example of a fictitious election. This is an important example, and we will revisit it many times throughout the chapter. You may want to think of Example 1.1 as a mathematical parable, its importance being not in the story itself but in what lies hidden behind it. (As you will soon see, there is a lot more to Example 1.1 than first meets the eye.)

## EXAMPLE 1.1 The Math Club Election (Winner-Only)

The Math Appreciation Society (MAS) is a student club dedicated to an unsung but worthy cause: that of fostering the enjoyment and appreciation of mathematics among college students. The MAS chapter at Tasmania State University is holding its annual election for club president, and there are four *candidates* running: Alisha, Boris, Carmen, and Dave ($A$, $B$, $C$, and $D$ for short).

Every member of the club is eligible to vote, and the vote takes the form of a *preference ballot*. Each voter is asked to rank each of the four candidates in order of preference. There are 37 *voters* who submit their ballots, and the 37 *preference ballots* submitted are shown in Fig. 1-2.

| Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1st *A* | 1st *B* | 1st *A* | 1st *C* | 1st *B* | 1st *C* | 1st *A* | 1st *B* | 1st *C* | 1st *A* | 1st *C* | 1st *D* | 1st *A* | 1st *A* | 1st *C* | 1st *A* | 1st *C* | 1st *D* |
| 2nd *B* | 2nd *D* | 2nd *B* | 2nd *B* | 2nd *D* | 2nd *B* | 2nd *B* | 2nd *D* | 2nd *B* | 2nd *B* | 2nd *B* | 2nd *C* | 2nd *B* | 2nd *B* | 2nd *B* | 2nd *B* | 2nd *B* | 2nd *C* |
| 3rd *C* | 3rd *C* | 3rd *C* | 3rd *D* | 3rd *C* | 3rd *D* | 3rd *C* | 3rd *C* | 3rd *D* | 3rd *C* | 3rd *D* | 3rd *B* | 3rd *C* | 3rd *C* | 3rd *D* | 3rd *C* | 3rd *D* | 3rd *B* |
| 4th *D* | 4th *A* | 4th *D* | 4th *A* | 4th *A* | 4th *A* | 4th *D* | 4th *A* | 4th *A* | 4th *D* | 4th *A* | 4th *A* | 4th *D* | 4th *D* | 4th *A* | 4th *D* | 4th *A* | 4th *A* |

| Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1st *C* | 1st *A* | 1st *D* | 1st *D* | 1st *C* | 1st *C* | 1st *D* | 1st *A* | 1st *D* | 1st *C* | 1st *A* | 1st *D* | 1st *B* | 1st *A* | 1st *C* | 1st *A* | 1st *A* | 1st *D* | 1st *A* |
| 2nd *B* | 2nd *B* | 2nd *C* | 2nd *C* | 2nd *B* | 2nd *B* | 2nd *C* | 2nd *B* | 2nd *C* | 2nd *B* | 2nd *B* | 2nd *C* | 2nd *D* | 2nd *B* | 2nd *D* | 2nd *B* | 2nd *B* | 2nd *C* | 2nd *B* |
| 3rd *D* | 3rd *C* | 3rd *B* | 3rd *B* | 3rd *D* | 3rd *D* | 3rd *B* | 3rd *C* | 3rd *B* | 3rd *D* | 3rd *C* | 3rd *B* | 3rd *C* | 3rd *C* | 3rd *B* | 3rd *C* | 3rd *C* | 3rd *B* | 3rd *C* |
| 4th *A* | 4th *D* | 4th *A* | 4th *A* | 4th *A* | 4th *A* | 4th *A* | 4th *D* | 4th *A* | 4th *A* | 4th *D* | 4th *A* | 4th *A* | 4th *D* | 4th *A* | 4th *D* | 4th *D* | 4th *A* | 4th *D* |

**Figure 1-2** The 37 preference ballots for the Math Club election.

Last but not least, what about the *outcome* of the election? Since the purpose of the election is to choose a club president, it is pointless to discuss or consider which candidate comes in second place, third place, etc. This is a *winner-only* election.

### EXAMPLE 1.2 The Math Club Election (Full Ranking)

Suppose now that we have pretty much the same situation as in Example 1.1 (same candidates, same voters, same preference ballots), but in this election we have to choose not only a president but also a vice-president, a treasurer, and a secretary. According to the club bylaws, the president is the candidate who comes in first, the vice-president is the candidate who comes in second, the treasurer is the candidate who comes in third, and the secretary is the candidate who comes in fourth. Given that there are four candidates, each candidate will get to be an officer, but there is a big difference between being elected president and being elected treasurer (the president gets status and perks; the treasurer gets to collect the dues and balance the budget). In this version how you place matters, and the outcome should be a full *ranking* of the candidates.

### EXAMPLE 1.3 The Academy Awards

2020 Academy Award winner (Best Actress) Renee Zellweger.

The Academy Awards (also known as the Oscars) are given out each year by the Academy of Motion Picture Arts and Sciences for Best Picture, Best Actress, Best Actor, Best Director, and many other categories (Sound Mixing, Makeup, etc.). The election process is not the same for all awards, so for the sake of simplicity we will just discuss the selection of Best Picture.

The *voters* in this election are all the eligible members of the Academy (approximately 8500 voting members in 2020). After a complicated preliminary round (a process that we won't discuss here), somewhere between eight and ten films are selected as the nominees—these are our *candidates*. (For most other awards there are only five nominees.) Each voter is asked to submit a preference ballot ranking all the candidates. There is only a winner (the other candidates are not ranked), with the winner determined by a voting method called plurality-with-elimination that we will discuss in detail in Section 1.4.

The part with which people are most familiar comes after the ballots are submitted and tabulated—the annual Academy Awards ceremony, held each year in late February. How many movie fans realize that behind one of the most extravagant and glamorous events in pop culture lies an election?

## EXAMPLE 1.4 The Heisman Trophy

The Heisman Memorial Trophy Award is given annually to the "most outstanding player in collegiate football." The Heisman, as it is usually known, is not only a very prestigious award but also a very controversial award. With so many players playing so many different positions, how do you determine who is the most "outstanding"?

2019 Heisman Trophy finalists Joe Burrow, Justin Fields, Jalen Hurts, and Chase Young.

In theory, any football player in any division of college football is a potential *candidate* for the award. In practice, the real candidates are players from Division I programs and are almost always in the glamour positions—quarterback or running back. (Since its inception in 1935, only once has the award gone to a defensive player—Charles Woodson of Michigan.)

The *voters* are members of the media plus all past Heisman award winners still living, plus one vote from the public (as determined by a survey conducted by ESPN). There are approximately 930 *voters* (the exact number of voters varies each year). Each voter submits a *truncated preference ballot* consisting of a first, second, and third choice (see Fig. 1-1[d]). A first-place vote is worth 3 points, a second-place vote 2 points, and a third-place vote 1 point. The candidate with the most total points from all the ballots is awarded the Heisman trophy in a televised ceremony held each December at the Downtown Athletic Club in New York. (We will discuss this voting method in more detail in Section 1.3.)

While only one player gets the award, the finalists are ranked by the number of total points received, in effect making the *outcome* of the Heisman trophy a *partial ranking* of the top candidates. (For the 2019 season, the winner was Joe Burrow of Louisiana State University, second place went to Jalen Hurts of Oklahoma, third place went to Justin Fields of Ohio State, and fourth place went to Chase Young, also of Ohio State.)

## EXAMPLE 1.5 American Idol

*American Idol* is a popular reality TV singing competition for individuals. Each year, the winner of *American Idol* gets a big recording contract, and many past winners have gone on to become famous recording artists (Kelly Clarkson, Carrie Underwood, Taylor Hicks). While there is a lot at stake and a big reward for winning, *American Idol* is not a winner-only competition, and there is indeed a ranking of all the finalists. In fact, some nonwinners (Clay Aiken, Jennifer Hudson) have gone on to become great recording artists in their own right.

The 10 candidates who reach the final rounds of the competition compete in a weekly televised show. During and immediately after each weekly show the voters cast their votes. The two candidates with the fewest number of votes are eliminated from the competition (sometimes when the voting is close only one candidate is eliminated), and the following week the process starts all over again with the remaining candidates. And who are the *voters* responsible for deciding the fate of these candidates? Anyone and everyone—you, me, Aunt Betsie—we are all potential voters. All one has to do to vote for a particular candidate is to go online, text or use an app. *American Idol* voting is an example of democracy run amok—you can vote for a candidate even if you never heard her sing, and you can vote as many times as you want.

By the final week of the competition the race is narrowed to the last three candidates, and after one last frenzied round of singing (followed by two more elimination rounds), the winner is determined. (See Example 1.16 for the full details on how it all played out in the 2019 American Idol finals.)

Examples 1.1 through 1.5 represent just a small sample of how elections can be structured, both in terms of the ballots (think of these as the *inputs* to the election) and the types of outcomes we look for (the *outputs* of the election). We will revisit some of these examples and many others as we wind our way through the chapter.

## Preference Ballots and Preference Schedules

Let's focus now on elections where the balloting is done by means of preference ballots, as in Examples 1.1 and 1.2. The great advantage of preference ballots (compared with, for example, single-choice ballots) is that they provide a great deal of useful information about an individual voter's preferences—in both direct and indirect ways.

To illustrate what we mean, consider the preference ballot shown in Fig. 1-3. This ballot directly tells us that the voter likes candidate $C$ best, $B$ second best, $D$ third best, and $A$ least. But, in fact, this ballot tells us a lot more—it tells us unequivocally which candidate the voter would choose if it came down to a choice between just two candidates. For example, if it came down to a choice between, say, $A$ and $B$, which one would this voter choose? Of course she would choose $B$—she has $B$ above $A$ in her ranking. Thus, a preference ballot allows us to make relative comparisons between any two candidates—*the candidate higher on the ballot is always preferred over the candidate in the lower position*. Please take note of this simple but important idea, as we will use it repeatedly later in the chapter.

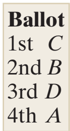



Figure 1-3

The second important idea we will use later is the assumption that the relative preferences in a preference ballot do not change if one of the candidates withdraws or is eliminated. Once again, we can illustrate this using Fig. 1-3. What would happen if for some unforeseen reason candidate $B$ drops out of the race right before the ballots are tabulated? Do we have to have a new election? Absolutely not—the old ballot simply becomes the ballot shown on the right side of Fig. 1-4. The candidates above $B$ stay put and each of the candidates below $B$ moves up a spot.

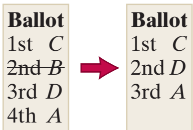



Figure 1-4

In an election with many voters, some voters will vote exactly the same way—for the same candidates in the same order of preference. If we take a careful look at the 37 ballots submitted for the Math Club election shown in Fig. 1-2, we see that 14 ballots look exactly the same ($A$ first, $B$ second, $C$ third, $D$ fourth), another 10 ballots look the same, and so on. So, if you were going to tabulate the 37 ballots, it might make sense to put all the $A$-$B$-$C$-$D$ ballots in one pile, all the $C$-$B$-$D$-$A$ ballots in another pile, and so on. If you were to do this you would get the five piles shown in Fig. 1-5 (the order in which you list the piles from left to right is irrelevant). Better yet, you can make the whole idea a little more formal by putting all the ballot information in a table such as Table 1-1, called the **preference schedule** for the election.

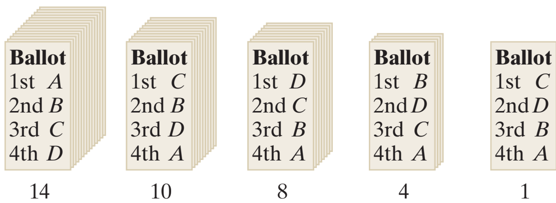



Figure 1-5 The 37 Math Club election ballots organized into piles.

| Number of voters | 14 | 10 | 8 | 4 | 1 |
|---|---|---|---|---|---|
| 1st | $A$ | $C$ | $D$ | $B$ | $C$ |
| 2nd | $B$ | $B$ | $C$ | $D$ | $D$ |
| 3rd | $C$ | $D$ | $B$ | $C$ | $B$ |
| 4th | $D$ | $A$ | $A$ | $A$ | $A$ |

Table 1-1 ■ Preference schedule for the Math Club election

We will be working with preference schedules throughout the chapter, so it is important to emphasize that a preference schedule is nothing more than a convenient bookkeeping tool—it summarizes all the elements that constitute the input to an election: the candidates, the voters, and the balloting. Just to make sure this is clear, we conclude this section with a quick example of how to read a preference schedule.

### EXAMPLE 1.6 The City of Kingsburg Mayoral Election

| Number of voters | 93 | 44 | 10 | 30 | 42 | 81 |
|---|---|---|---|---|---|---|
| 1st | *A* | *B* | *C* | *C* | *D* | *E* |
| 2nd | *B* | *D* | *A* | *E* | *C* | *D* |
| 3rd | *C* | *E* | *E* | *B* | *E* | *C* |
| 4th | *D* | *C* | *B* | *A* | *A* | *B* |
| 5th | *E* | *A* | *D* | *D* | *B* | *A* |

**Table 1-2** ■ Preference schedule for the Kingsburg mayoral election

Table 1-2 shows the preference schedule summarizing the results of the most recent election for mayor of the city of Kingsburg (there actually is a city by that name, but the election is fictitious). Just by looking at the preference schedule we can answer all of the relevant input questions:

- *Candidates*: there were five candidates ($A$, $B$, $C$, $D$, and $E$, which are just abbreviations for their real names).
- *Voters*: there were 300 voters that submitted ballots (add the numbers at the head of each column: $93 + 44 + 10 + 30 + 42 + 81 = 300$).
- *Balloting*: the 300 preference ballots were organized into six piles as shown in Table 1-2.

The question that still remains unanswered: Who is the winner of the election? In the next four sections we will discuss different ways in which such output questions can be answered.

## Ties

In any election, be it a *winner-only* election or a *ranking* of the candidates, ties can occur. What happens then?

In some elections (for example, Academy Awards, many sports awards, and reality TV competitions) ties are allowed to stand and need not be broken. Here are a few interesting examples:

- **Academy Awards:** In 1932, Frederic March and Wallace Beery tied for Best Actor; in 1969, Katharine Hepburn and Barbra Streisand tied for Best Actress. A few more ties for lesser awards have occurred over the years. When ties occur, both winners receive the Oscar.
- **Grammys:** Over the years, there have been several ties for Grammy Awards. The most recent: A tie for Best Rap Performance at the 2019 Grammys between Anderson .Paak (for "Bubblin") and Jay Rock, Kendrick Lamar, Future, and James Blake (for "King's Dead").

2019 Grammy tie for Best Rap Performance: Jay Rock (left); Anderson .Paak (right).

- **NFL Most Valuable Player:** In 2004, Peyton Manning and Steve McNair shared the MVP award. (So did Brett Favre and Barry Sanders in 1997, as well as Norm van Brocklin and Joe Schmidt in 1960.)
- **Cy Young Award:** There has only been one tie in the history of the Cy Young Award. In 1969 Mike Cuellar of the Baltimore Orioles and Denny McLain of the Detroit Tigers tied for the American League award in 1969. There have been no other ties since.

In other situations, especially in elections for political office (president, senator, mayor, city council, etc.), ties cannot be allowed (can you imagine having co-presidents or co-mayors?), and then a tie-breaking rule must be specified. The Constitution, for example, stipulates how a tie in the Electoral College is broken, and most elections have a set rule for breaking ties. The most common method for breaking a tie for political office is through a runoff election, but runoff elections are expensive and take time, so many other tie-breaking procedures are used. Here are a few interesting examples:

- In the 2018 election for a seat in the Virginia House of Delegates' 94th District, incumbent Republican David Yancey received 11,608 votes. Problem was that his opponent, Democrat Shelly Simonds also received 11,608 votes and there can only be one winner, so what to do? A 1705 Virginia law to the rescue: ties must be broken by drawing names "out of a hat." (The "hat" turned out to be a figure of speech—each of the names was placed inside a small canister and the canisters put inside a ceramic bowl.) When all was said and done, Mr. Yancey retained his seat as his name was drawn first. The moral of this story is that "every vote counts" is not just a cliche.

- In the 2009 election for a seat in the Cave Creek, Arizona, city council, Thomas McGuire and Adam Trenk tied with 660 votes each. The winner was decided by drawing from a deck of cards. Mr. McGuire drew first—a six of hearts. Mr. Trenk (the young man with the silver belt buckle) drew next and drew a king of hearts. This is how Mr. Trenk became a city councilman.

Ties and tie-breaking procedures add another layer of complexity to an already rich subject. To simplify our presentation, in this chapter we will stay away from ties as much as possible.

## 1.2 The Plurality Method

The **plurality method** is arguably the simplest and most commonly used method for determining the outcome of an election. With the plurality method, all that matters is how many first-place votes each candidate gets: In a *winner-only* election the candidate with the most first-place votes is the winner; in a *ranked* election the candidate with the most first-place votes is first, the candidate with the second most is second, and so on.

For an election decided under the plurality method, *preference ballots* are not needed, since the voter's second, third, etc. choices are not used. But, since we already have the preference schedule for the Math Club election (Examples 1.1 and 1.2) let's use it to determine the outcome under the plurality method.

### EXAMPLE 1.7 The Math Club Election Under the Plurality Method

| Number of voters | 14 | 10 | 8 | 4 | 1 |
|---|---|---|---|---|---|
| 1st | *A* | *C* | *D* | *B* | *C* |
| 2nd | *B* | *B* | *C* | *D* | *D* |
| 3rd | *C* | *D* | *B* | *C* | *B* |
| 4th | *D* | *A* | *A* | *A* | *A* |

**Table 1-3** ■ Preference schedule for the Math Club election

We discussed the Math Club election in Section 1.1. Table 1-3 shows once again the preference schedule for the election. Counting only first-place votes, we can see that *A* gets 14, *B* gets 4, *C* gets 11, and *D* gets 8. So there you have it: In the case of a *winner-only* election (see Example 1.1) the winner is *A* (Headline: "Alisha wins presidency of the Math Club"); in the case of a *ranked election* (see Example 1.2) the results are: *A* first (14 votes); *C* second (11 votes); *D* third (8 votes); *B* fourth (4 votes). (Headline "New board of MAS elected! President: Alisha; VP: Carmen; Treasurer: Dave; Secretary: Boris.")